Variable free reasoning on finite trees

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Introduction

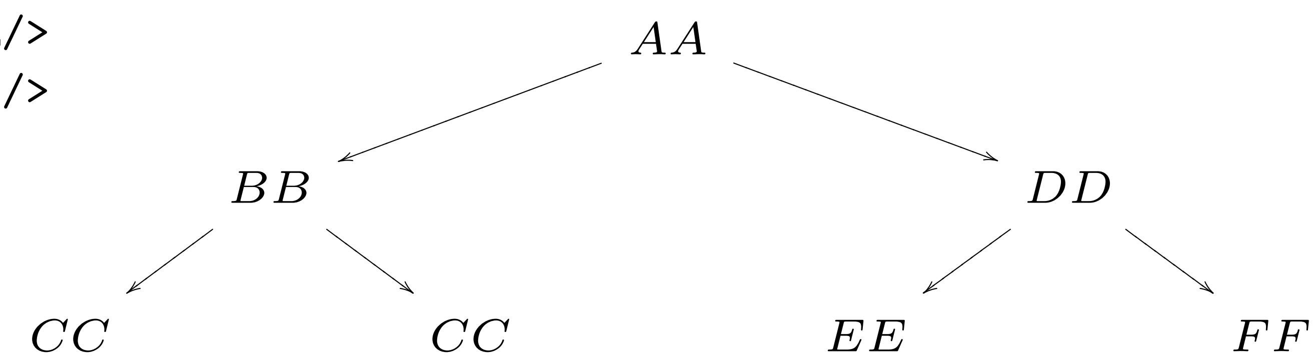
- Modal languages interpreted on finite trees.
- More precisely, node labelled, sibling ordered finite trees.
- Interested in *Completeness* questions:
  - Functional completeness
  - Completeness for definitions (Beth's property)
  - Complete optimal decison algorithms

Motivation

- New data storage format: XML.
- XML documents are finite node labeled ordered trees.
- Most succesful XML query language XPath is variable free, and very modal in flavour.
- XPath query containment can effectively be reduced to satisfiability in a corresponding modal language.
- XPath query evaluation can be reduced to model checking for the corresponding modal language.

XML documents are node labelled ordered trees

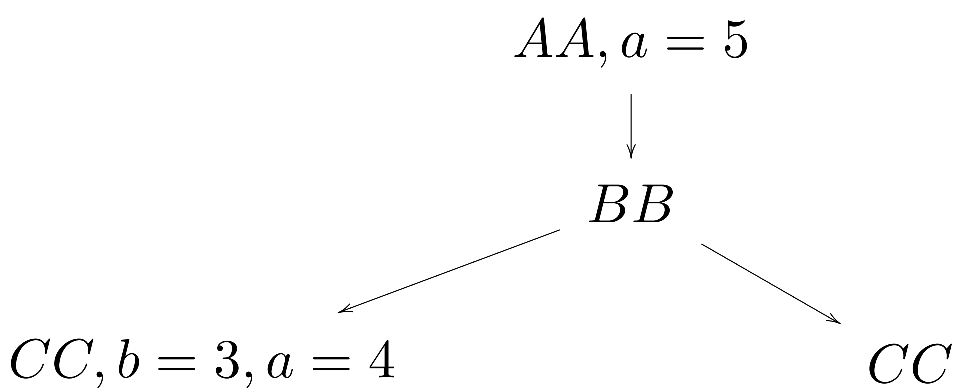
```
<AA>
  <BB>
    <CC/>
    <CC/>
  </BB>
  <DD>
    <EE/>
    <FF/>
  </DD>
</AA>
```



Attributes

XML Attributes can be modelled by multiple labels.

```
<AA a=5>
  <BB>
    <CC b=3,a=4/>
    <CC/>
  </BB>
</AA>
```

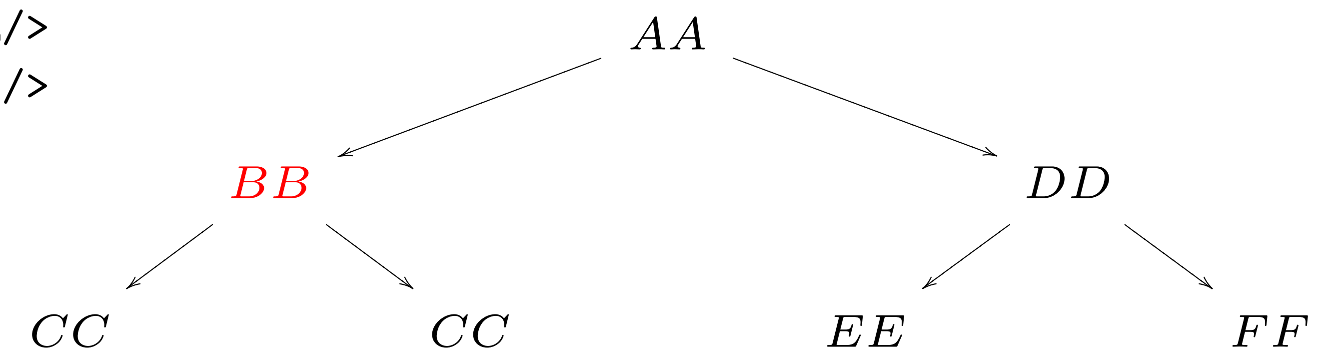


XPath, a query language for XML

- XPath is a W3C standard query language for XML documents.
- Given a XML document D, a node n in D, an XPath query Q selects all nodes from D which are reachable from n by the path described in Q.
- Examples:

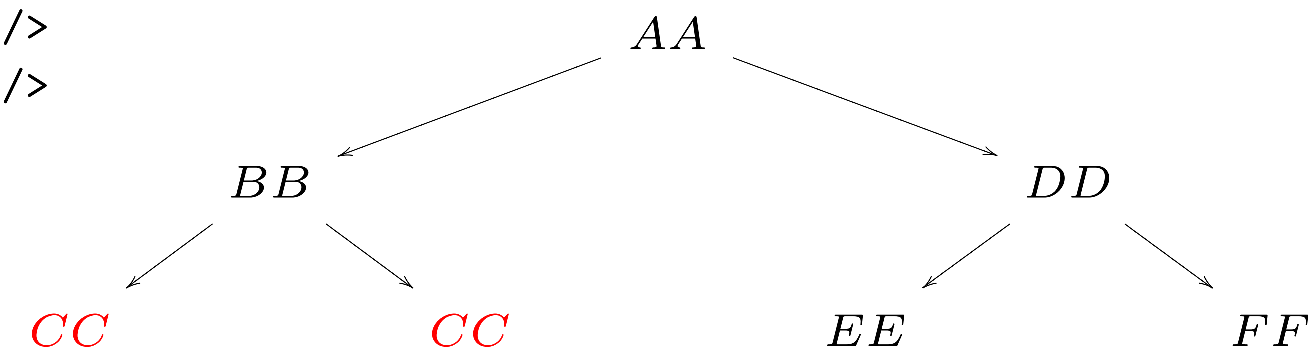
/AA/BB

```
<AA>
  <BB>
    <CC/>
   <CC/>
  </BB>
  <DD>
    <EE/>
    <FF/>
  </DD>
</AA>
```



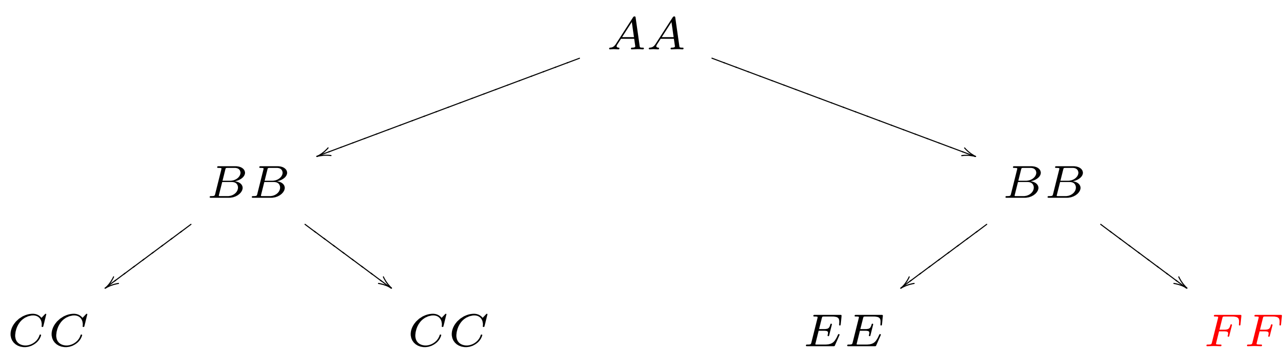
//CC

```
<AA>
  <BB>
   <CC/>
   <CC/>
  </BB>
  <DD>
    <EE/>
    <FF/>
  </DD>
</AA>
```



//BB[EE]/FF

```
<AA>
  <BB>
    <CC/>
    <FF/>
  </BB>
  <BB>
   <EE/>
   <FF/>
  </BB>
</AA>
```



Finite sibling ordered trees

- First order structures with two binary relations.
- Domain is a finite set.
- Dominance relation:

$$xR_{\downarrow}y \text{ iff } x \text{ is an ancestor of } y.$$

- Linear order on the children of each node

$$xR_{\rightarrow}y \text{ iff } x \text{ and } y \text{ are siblings and } x \text{ is strictly on the left of } y.$$

- First order language in signature $R_{\downarrow}$ and $R_{\rightarrow}$ and unary $P_1, P_2, \ldots$

Modal or variable-free approaches

The language is two-sorted with interactions:

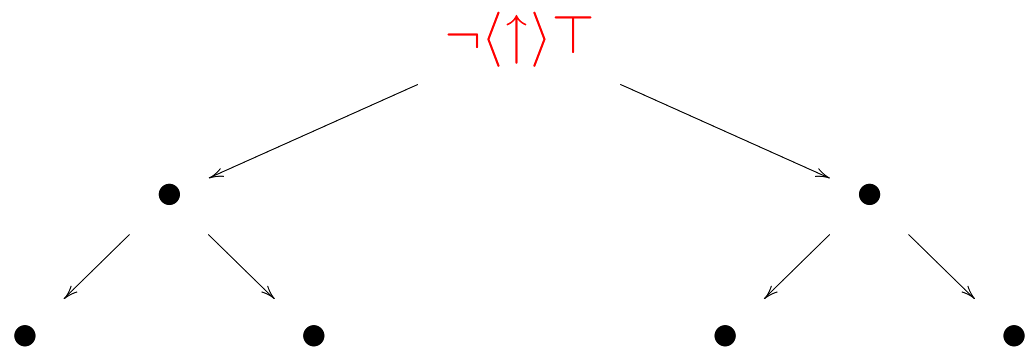
- Sort for paths in the tree: regular expressions over the four basic steps
  - $\downarrow$ (child),
  - $\uparrow$ (parent),
  - $\rightarrow$ (right sibling),
  - $\leftarrow$ (left sibling).
- Sort for nodes in the tree:
  - labels
  - closed under the Boolean operations

Interactions:

- if π is of the path sort, and ϕ of the node sort, then $\langle\pi\rangle\phi$ is of the node sort.
  - $\langle\pi\rangle\phi$ holds at nodes from which there is a π path to a ϕ node.
  - $V(\langle\pi\rangle\phi) = \{t \mid \exists t' : t \; \pi \; t' \wedge t' \in V(\phi)\}$.
- if ϕ is of the node sort, then $?\phi$ is of the path sort.
  - $?\phi$ is called a test.
  - $?\phi$ denotes the identity path from a ϕ node to itself.
  - $?\phi$ denotes $\{(t,t) \mid t \in V(\phi)\}$.

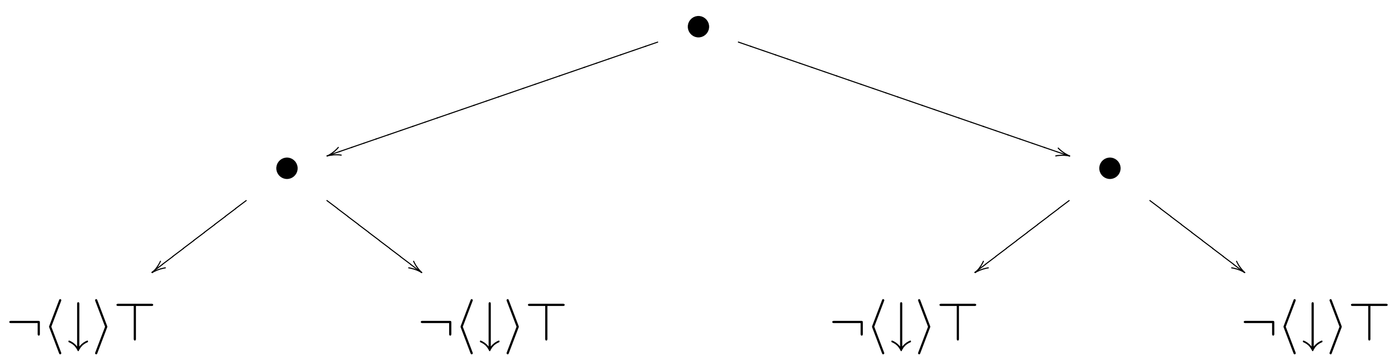
Example expressions

I am the root: $\neg\langle\uparrow\rangle\top$



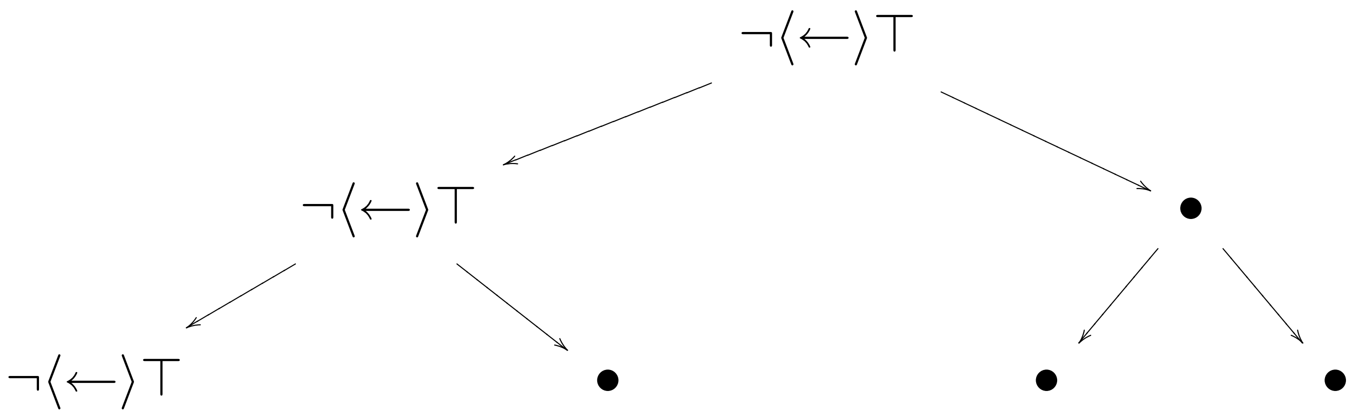
Example expressions

I am a leaf: $\neg\langle\downarrow\rangle\top$



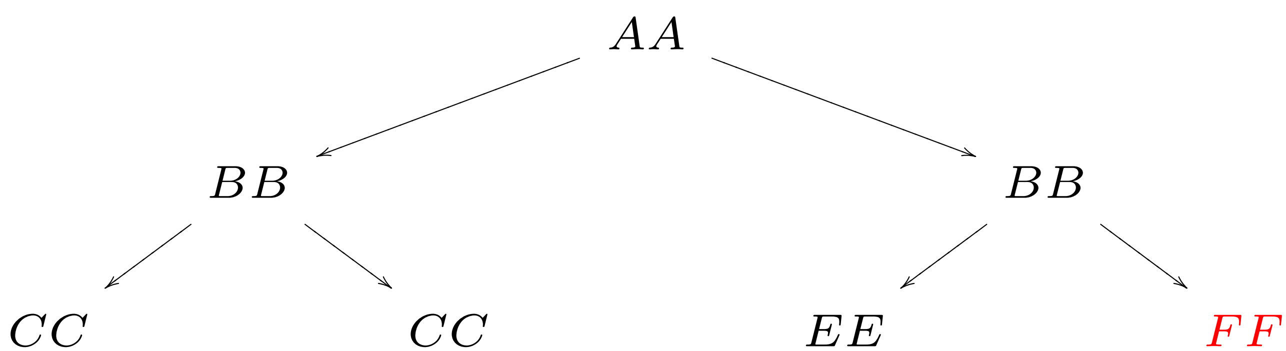
Example expressions

I am a first daugther: $\neg\langle\leftarrow\rangle\top$



//BB[EE]/FF

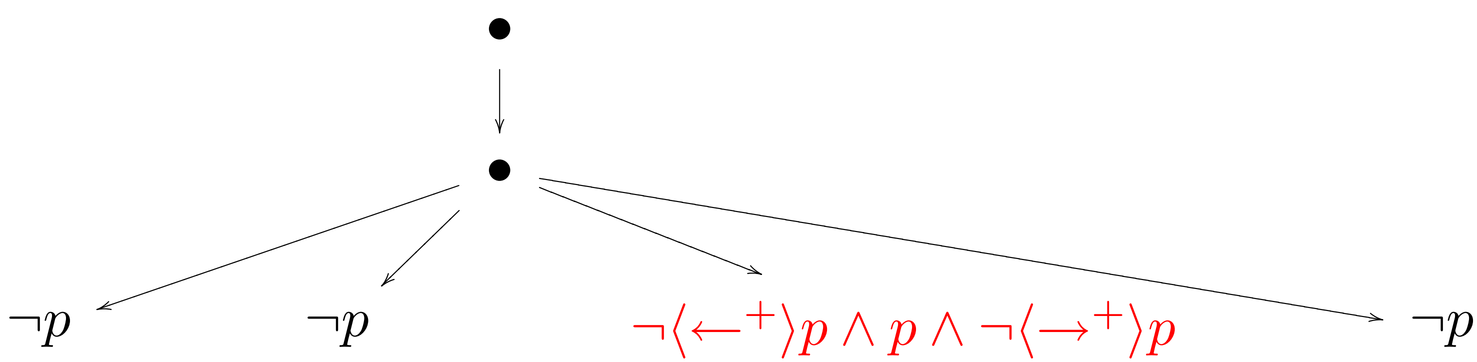
```
<AA>
  <BB>
    <CC/>
    <FF/>
  </BB>
  <BB>
   <EE/>
   <FF/>
  </BB>
</AA>
```



//BB[EE]/FF is equivalent to $FF \wedge \langle\uparrow\rangle(BB \wedge \langle\downarrow\rangle EE \wedge \langle\uparrow^*\rangle root)$.

Example expressions

I am the unique p among my siblings: $p \wedge \neg\langle\leftarrow^{+}\rangle p \wedge \neg\langle\rightarrow^{+}\rangle p$



Three languages

- Full PDL. Here called $\mathcal{X}_{Reg}$. [Kracht 95]

$$\begin{array}{lcl} \pi & ::= & \leftarrow \;|\; \rightarrow \;|\; \uparrow \;|\; \downarrow \;|\; \pi;\pi \;|\; \pi \cup \pi \;|\; \pi^* \;|\; ?\phi \\ \phi & ::= & p \;|\; \top \;|\; \neg\phi \;|\; \phi \wedge \phi \;|\; \langle\pi\rangle\phi. \end{array}$$

- Fragment of $\mathcal{X}_{Reg}$, keeping *conditional paths*. Here called $\mathcal{X}_{cp}$. [Palm 95]

$$\pi ::= \leftarrow \;|\; \rightarrow \;|\; \uparrow \;|\; \downarrow \;|\; ?\phi;\pi \;|\; \pi^*.$$

- Modal language corresponding to Core XPath: $\mathcal{X}_{Core}$.

$$\pi ::= \leftarrow \;|\; \rightarrow \;|\; \uparrow \;|\; \downarrow \;|\; \pi^*.$$

Overview

1. Comparing expressive power. (functional completeness)
2. Completeness for definitions (Beth's property)
3. Model checking
4. Complete decision algorithms

Comparing expressive power

- How are these three languages related?
- How do they relate to other formalisms used on ordered trees?

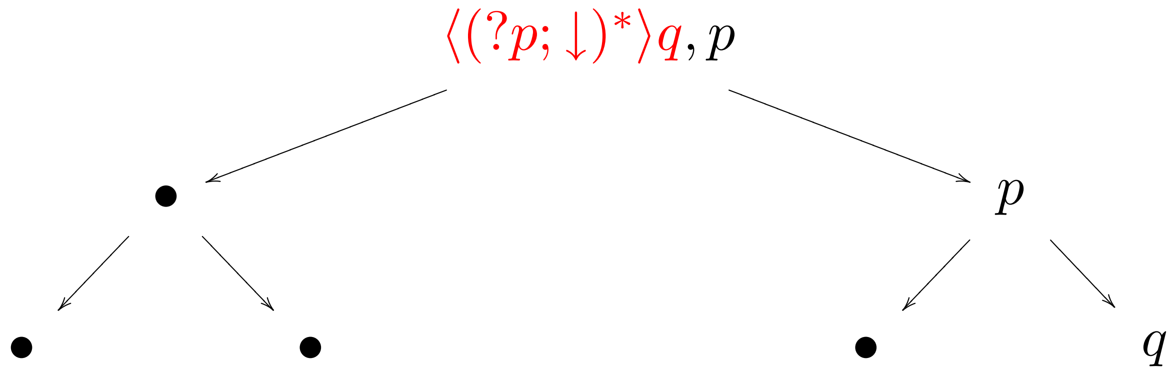
1. $\mathcal{X}_{Core} \subsetneq \mathcal{X}_{cp} \subsetneq \mathcal{X}_{Reg}$.
2. $\mathcal{X}_{cp}$ is equivalent to the language with just four until operators.
3. $\mathcal{X}_{cp}$ is first order logic and $\mathcal{X}_{Reg}$ is stronger.

$\mathcal{X}_{Core}$ and Core XPath

- We study the W3C standard XPath 1.0.
- Gottlob et al singled out the *logical core* of XPath 1.0, and called it Core XPath.
- Theorem Every Core XPath root expression is equivalent to an $\mathcal{X}_{Core}$ expression.
- E.g., `/AA//BB` is equivalent to $BB \wedge \langle\uparrow^*\rangle(AA \wedge root)$.

Adding conditional paths

- A conditional path is an expression of the form $?\phi; \pi$.
- **EXAMPLE** $\langle(?p; \downarrow)^*\rangle q$ is true on all nodes from which there is a path going down along p nodes ending in a q node.



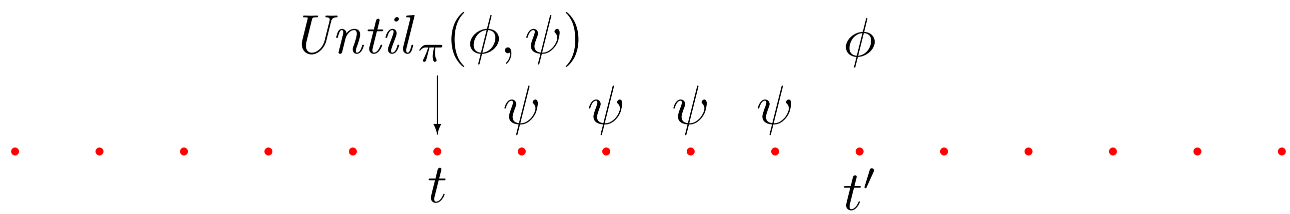
- $\langle(?p; \downarrow)^*\rangle q$ behaves like Until in temporal logic: Until q holds, p is true.

$\mathcal{X}_{cp}$ and until

- $\mathcal{X}_{until}$ is the modal language with four binary modal operators: $Until_\pi(\phi, \psi)$ for $\pi \in \{\leftarrow, \rightarrow, \uparrow, \downarrow\}$,

- $\mathfrak{M}, t \models Until_\pi(\phi, \psi)$ iff

$$\exists t'(t\ \pi^+\ t' \wedge \mathfrak{M}, t' \models \phi \wedge \forall t''(t\ \pi^+ t''\ \pi^+\ t' \rightarrow \mathfrak{M}, t'' \models \psi)).$$



- Theorem $\mathcal{X}_{cp}$ and $\mathcal{X}_{until}$ are equally expressive.

$\mathcal{X}_{cp}$ and first order logic

- By the meaning definition of $Until_{\pi}(\phi, \psi)$, $\mathcal{X}_{until}$ is contained in the first order logic of trees.

- By Kamp's Theorem, on linear trees, the reverse also holds.

- Gabbay strenghtened Kamp's theorem into the separation theorem: every $\mathcal{X}_{until}$ formula interpreted in linear trees is equivalent to a boolean combination of past, future and present formulas.

- We generalized Gabbay's result to all ordered trees.

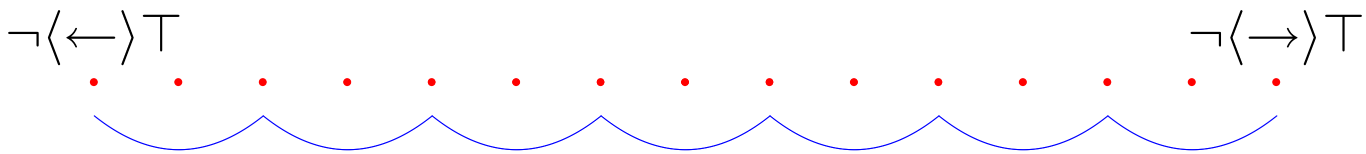
- Thus $\mathcal{X}_{cp}$, $\mathcal{X}_{until}$ and first order logic are equally expressive.

Functional Completeness

Theorem Every first order definable set of nodes in an ordered tree is definable by an $\mathcal{X}_{cp}$ (or equivalently, by an $\mathcal{X}_{until}$) formula.

But $\mathcal{X}_{Reg}$ is more expressive. It can express second order properties of nodes like having an odd number of daughters:

$$\langle\downarrow\rangle(\neg\langle\leftarrow\rangle\top \wedge \langle(\rightarrow;\rightarrow)^*\rangle\neg\langle\rightarrow\rangle\top).$$



Overview

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2. Completeness for definitions (Beth's property)

3. Model checking

4. Complete decision algorithms

Beth's property

- Does not hold for the first order (until) fragment.
- Let Γ be $root \rightarrow p \;\; \wedge \;\; p \rightarrow [\downarrow]\neg p \;\; \wedge \;\; \neg p \rightarrow [\downarrow]p$.
- For $\mathfrak{M}$ any tree, if $\mathfrak{M} \models \Gamma$ then $\mathfrak{M}, n \models p \iff n$ is even.
- But evenness is not first order expressible.
- The $\mathcal{X}_{Reg}$ definition of p is of course $\langle(\uparrow;\uparrow)^*\rangle root$.

Question 1 Can we find for all implicit $\mathcal{X}_{cp}$ definitions, the explicit definition in $\mathcal{X}_{Reg}$? Or better, for interpolants?

Question 2 Does $\mathcal{X}_{Reg}$ have interpolation or Beth's property? (for PDL this is still unknown).

Overview

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Model checking

- XPath query evaluation can effectively be reduced to $\mathcal{X}_{Reg}$ model checking.

- From [Gottlob et al, PODS 2003] and the equivalence between $\mathcal{X}_{Core}$ and Core XPath it follows that $\mathcal{X}_{Core}$ model checking is hard for **PTIME** (combined complexity).

- The largest language $\mathcal{X}_{Reg}$ can be model checked in linear time, that is, $O(|D| \cdot |Q|)$.

- This follows from the same result for PDL.

Overview

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Deciding $\mathcal{X}_{Reg}$

- We want to decide the question: given a set Γ of constraints on trees, is ϕ valid on these trees?
- Or $\Gamma \models \phi$, where $\models$ is the *global* consequence relation.
- Motivation
  - Query optimization
  - DTD's XSchema

Deciding $\mathcal{X}_{Reg}$, Complexity

- Decidability is easily obtained from Rabin's theorem, but the complexity is too high.

- Easy lower bound is EXPTIME (by an interpretation of ordinary PDL with one program and only the diamonds $\langle a \rangle$ and $\langle a^* \rangle$.

Decision algorithm by reduction

- Goal: Effectively reduce $\phi \models \psi$ to $\phi' \models \psi'$ in a simpler language.
- In fact to PDL without complex programs at all, but only with the paths $\downarrow$ and $\rightarrow$.
- Then to modal logic of finite binary branching trees.
- Consequence problem for this can easily be shown to be in EXPTIME.
  - reduce it to CTL plus counting, or
  - by bottom up Hintikka Set Elimination.

Reductions

- Idea 1 Delete union, composition and tests by the PDL equivalences:
- $\langle \pi_1 ; \pi_2 \rangle \phi \equiv \langle \pi_1 \rangle \langle \pi_2 \rangle \phi$.
- $\langle \pi_1 \cup \pi_2 \rangle \phi \equiv \langle \pi_1 \rangle \phi \vee \langle \pi_2 \rangle \phi$.
- $\langle ?\psi \rangle \phi \equiv \phi \wedge \psi$.

Reductions (for star)

- Idea 2 "Axiomatize" starred programs:
- for each subformula θ add a new propositional variable q_θ.
- Replace θ throughout by q_θ.
- Ensure that $\theta \equiv q_\theta$ by a constraint.

Reductions for star using constraints

- Example $\langle\downarrow^*\rangle p$ Add as a constraint:

$$q_{\langle\downarrow^*\rangle p} \leftrightarrow p \vee \langle\downarrow\rangle q_{\langle\downarrow^*\rangle p}. \tag{1}$$

- Then $(1) \models q_{\langle\downarrow^*\rangle p} \leftrightarrow \langle\downarrow^*\rangle p$.

- Proof is by induction on the (finite!) number of daughters.

  - t is a leaf:

$$\begin{array}{llr} t \models q_{\langle\downarrow^*\rangle p} & \Longleftrightarrow & \text{(by (1))} \\ t \models p \vee \langle\downarrow\rangle q_{\langle\downarrow^*\rangle p} & \Longleftrightarrow & \text{(as } t \text{ is a leaf)} \\ t \models p & \Longleftrightarrow & \text{(by meaning def and } t \text{ is a leaf)} \\ t \models p \vee \langle\downarrow\rangle\langle\downarrow^*\rangle p & \Longleftrightarrow & \text{(by meaning def)} \\ t \models \langle\downarrow^*\rangle p. & & \end{array}$$

  - t has $k+1$ daughters: By induction hypothesis.

Where this technique breaks

- It is crucial for the last proof that the path under the star "makes a step".

- This is not the case with formulas of the following form
  - $\langle (\downarrow^*)^* \rangle \phi$
  - $\langle (\downarrow ; \uparrow)^* \rangle \phi$

- In fact the reduction does not work in these cases.

Counterexample

- Consider the not satisfiable formula $\langle(\downarrow^*)^*\rangle\bot$.
- Then the axiomatization becomes

$$
\begin{array}{lcl}
q_{\bot} & \leftrightarrow & \bot \\
q_{\langle(\downarrow^*)^*\rangle\bot} & \leftrightarrow & q_{\bot} \vee q_{\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot} \\
q_{\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot} & \leftrightarrow & q_{\langle(\downarrow^*)^*\rangle\bot} \vee q_{\langle\downarrow\rangle\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot} \\
q_{\langle\downarrow\rangle\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot} & \leftrightarrow & \langle\downarrow\rangle q_{\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot}.
\end{array}
$$

- $q_{\langle(\downarrow^*)^*\rangle\bot}$ can be satisfied in the trivial tree with only a root by setting the valuation

$$V(root) = \{q_{\langle\downarrow^*\rangle\langle(\downarrow^*)^*\rangle\bot}, q_{\langle(\downarrow^*)^*\rangle\bot}\}$$

- This model makes the axioms true.
- Thus the reduction does not work for the formula $\langle(\downarrow^*)^*\rangle\bot$.

Reduction for until

- The reduction works for the until language. The constraint is

$$q_{Until_\pi(\phi,\psi)} \leftrightarrow \langle\pi\rangle q_\phi \vee \langle\pi\rangle q_{(\psi\wedge Until_\pi(\phi,\psi))}.$$

- Effective reduction of $\phi \models \psi$ to $q_\phi, \nabla(\phi,\psi) \models q_\psi$, with $\nabla(\phi,\psi)$

- Right hand only contains diamonds of the form $\langle\pi\rangle$ for $\pi \in \{\downarrow,\uparrow,\leftarrow,\rightarrow\}$.

- Reduce it further to $\phi \models \psi$ on binary branching trees for formulas containing only diamonds $\langle\downarrow_0\rangle$ and $\langle\downarrow_1\rangle$.

Modal logic of binary trees

- The consequence problem is different for finite and for infinite trees:
- For instance, $\langle \downarrow_0 \rangle \top \models \bot$ is true on finite trees, but not on all trees.
- For this reason we cannot use known results about deterministic PDL or ordinary modal logic.
- But we can embed the problem into CTL plus counting.

Mimicking finiteness in CTL

- Relativize all diamonds by a new variable d.
- Add as constraints:
  - d holds at the root.
  - for every path there is a point in the future where d is false.
  - Everywhere, if d is false it remains false forever.
- Then $\mathfrak{M}$ relativized to $[d]_{\mathfrak{M}}$ is a finite tree.

Compare with first order logic on trees

- Recall that the until language and the first order language are equally expressive on ordered trees.
- The until language is decidable in EXPTIME.
- The optimal decision procedure for the first order language is non-elementary!

Conclusions and further research

- Variable free tree formalisms are to be preferred over first order formalisms:
  - formulas are easy and intuitive
  - the computational complexity is much lower while having the same expressive power.

- Current work focuses on rewriting $\mathcal{X}_{Reg}$ formulas into a normal form which can be reduced.

- We conjecture that the EXPTIME result holds for the full orientation logic.

- Further research is needed on Beth's definability property.

- Also desirable: a natural extension of first order logic which is as expressive as $\mathcal{X}_{Reg}$.